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Self-adjoint acoustic equations with progressing wave solutions

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Abstract. The simplest wave equations are those whose general solutions comprise progressing waves. We construct a comparatively large, possibly exhaustive, family of self-adjoint acoustic equations in 1+1 dimensions that are simple in this sense.

1. Introduction

Partial differential equations with variable coefficients that are exactly solvable in some useful sense are an important exception to the general case. In this paper we shall restrict ourselves to equations in 1+1 dimensions of the special form

$$c^2(x, t)\, W_{xx} = W_{tt} \tag{1.1}$$

usually referred to as *acoustic equations*, and further restrict ourselves to examples of (1.1) that are self-adjoint. We shall construct a family of such equations all of whose members are exactly solvable in a sense to be made precise, solvable self-adjoint acoustic equations (SSAE), and which may include all such cases.

We shall consider a linear wave equation to be *exactly solvable* when its general solution is a finite sum of progressing waves of finite order [1, 2]. In 1+1 dimensions a progressing wave is a function that can be written in the form

$$W = \sum_{n=0}^{N} h_n(u, v) \frac{\mathrm{d}^n S(z)}{\mathrm{d}z^n} \tag{1.2}$$

where the $h_n(u, v)$ are fixed functions of the characteristic coordinates u and v, and z is either u or v, with S any sufficiently differentiable function. Clearly an equation of the type (1.1) will be exactly solvable precisely when its general solution takes the form

$$W = \sum_{n=0}^{N} g_n(u, v) \frac{\mathrm{d}^n a(u)}{\mathrm{d}u^n} + \sum_{n=0}^{N'} f_n(u, v) \frac{\mathrm{d}^n b(v)}{\mathrm{d}v^n} \tag{1.3}$$

a trivial example is provided by

$$W_{xx} = W_{tt} \qquad W = a(x-t) + b(x+t). \tag{1.4}$$

It is apparent that any such equation is mathematically simple since its general solution can be written without recourse to infinite series or non-trivial integrals. It is physically simple in the sense that given any pair of characteristics $u = u_0$, $u = u_1$, or, alternatively, $v = v_0$, $v = v_1$, there exists a solution determined by an arbitrary function of one variable, whose support is bounded by that pair of characteristics. This behaviour

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has been called 'characteristic propagation' [3], and the corresponding solutions are variously referred to as 'tailless' [3], 'non-spreading' [4], or 'wake-free' [5]. In general relativity such equations are associated with 'transparent spacetimes' [6], and in quantum mechanics with 'reflectionless potentials' [7]. These references indicate the variety of settings in which it has been found worthwhile to search for exactly solvable wave equations. Exactly solvable acoustic equations in $1+1$ dimensions are the subject of a recent publication [8], and our construction of possibly all such equations among those that are self-adjoint significantly extends some of the results of that paper. It is worth noting that even a single example essentially different from (1.4) is not immediately evident.

A sufficient and probably necessary condition for any homogeneous linear second-order wave equation in $1+1$ dimensions to be exactly solvable in our sense was given in 1968 by Kundt and Newman [3]. A feature of their approach is that it includes an explicit construction of the progressing wave general solution. Their criterion applies to the normal form

$$[\partial_v j_0(u, v)\partial_u - j_1(u, v)]\psi_0 = 0 \tag{1.5}$$

into which every such wave equation can always be transformed [3]. Equation (1.5) is explicitly self-adjoint when $j_0(u, v) = 1$. It is easy to see that transformations of the form $\bar{u} = \bar{u}(u)$, $\bar{v} = \bar{v}(v)$ and $\bar{\psi}(u, v) = k(v)\psi_0(u, v)$, preserve the normal form (1.5), which means that

$$j_0(u, v) = m(u)l(v) \tag{1.6}$$

characterises the essentially self-adjoint equations. Thus our first goal is to isolate those functions $c(x, t)$ for which (1.1), when transformed into the form (1.5), yields (1.6). The result is quite simple, as is the special class of $j_1(u, v)$ that can result. Although the calculation is elementary we have no reference for it, so the details are given in section 2.

Once all the self-adjoint acoustic equations have been expressed in normal form one wishes to specialise the $j_1(u, v)$ further to those for which (1.5) is in fact exactly solvable. Fortunately, results obtained recently [9, 10] provide an explicit construction of *all* pairs $j_0(u, v)$, $j_1(u, v)$ for which the Kundt–Newman criterion is satisfied, and an equally explicit construction of all the $j_1(u, v)$ for which that criterion is satisfied and $j_0(u, v) = 1$. The reader is referred to [9, 10] for details; in section 3 we restate the results from [9] that we shall need here.

In section 4 we specialise the results stated in section 3 to the particular class of $j_1(u, v)$ obtained in section 2. This specialisation is not trivial; however, a clean solution can be obtained providing a relatively large class of SSAE and their progressing wave general solutions. It seems to us likely that the family constructed includes all the SSAE, but that is not proved here.

In section 5 we work out the details of a special case to illustrate our result, and relate it to results in [8]. We also discuss the possible extension of this work to the non-self-adjoint case.

2. Self-adjoint acoustic equations

If we apply the transformation

$$u = u(x, t) \qquad v = v(x, t) \tag{2.1}$$

to (1.1) we obtain

$$2(c^2u_xv_x - u_tv_t)W_{uv} + (c^2u_{xx} - u_{tt})W_u + (c^2v_{xx} - v_{tt})W_v = 0 \tag{2.2}$$

precisely when the transformation satisfies $c^2(x, t)u_x^2 = u_t^2$, $c^2(x, t)v_x^2 = v_t^2$. It follows that $c(x, t)u_x = \pm u_t$ and $c(x, t)v_x = \pm v_t$, and in order that the Jacobian of (2.1) be different from zero we must choose opposite signs in these equations. Without loss of generality we choose

$$c(x, t)u_x = -u_t \qquad c(x, t)v_x = +v_t. \tag{2.3}$$

However, taking the x and t derivatives of each of (2.3) and using the equality of mixed partial derivatives yields

$$c^2u_{xx} = u_{tt} = (c_t - cc_x)u_x \qquad c^2v_{xx} - v_{tt} = -(c_t + cc_x)v_x \tag{2.4}$$

so (2.2) becomes

$$W_{uv} + AW_u + BW_v = 0 \qquad A \equiv \frac{c_t - cc_x}{4c^2v_x} \qquad B \equiv -\frac{c_t + cc_x}{4c^2u_x}. \tag{2.5}$$

The normal form (1.5) with $\psi = \Sigma(u, v)W$ becomes

$$W_{uv} + \frac{\Sigma_u}{\Sigma}W_v + \left(\frac{\Sigma_v}{\Sigma} + (\ln|j_0|)_v\right)W_u = 0 \tag{2.6}$$

precisely when the transformation function Σ is any solution of (1.5). Equations (2.5) and (2.6) are identical when

$$\frac{\Sigma_u}{\Sigma} = B \qquad \frac{\Sigma_v}{\Sigma} + (\ln|j_0|)_v = A. \tag{2.7}$$

However, (1.5) is self-adjoint just when we can choose $j_0 = 1$, in which case (2.7) becomes

$$\Sigma_u/\Sigma = B \qquad \Sigma_v/\Sigma = A. \tag{2.8}$$

It is immediate that

$$\begin{aligned} (\ln|\Sigma|)_x &= Bu_x + Av_x = -(\ln\sqrt{c})_x \\ (\ln|\Sigma|)_t &= Bu_t + Av_t = +(\ln\sqrt{c})_t \end{aligned} \tag{2.9}$$

and from this it follows that

$$(\ln|\Sigma|)_{xt} = -(\ln\sqrt{c})_{xt} = +(\ln\sqrt{c})_{xt} \tag{2.10}$$

which implies that

$$(\ln\sqrt{c})_{xt} = 0 \qquad c(x, t) = f(t)/g(x) \tag{2.11}$$

where f and g are arbitrary functions of one variable.

If we *start* with (2.11) then from (2.9) we have

$$\begin{aligned} (\ln\Sigma)_x + (\ln|j_0|)_v v_x &= (\ln g)_x \\ (\ln\Sigma)_t + (\ln|j_0|)_v v_t &= (\ln f)_t \end{aligned} \tag{2.12}$$

which imply, after differentiating with respect to t and x, respectively, and subtracting, that

$$(\ln|j_0|)_{vt}v_x = (\ln|j_0|)_{vx}c(x, t)v_x. \tag{2.13}$$

Given (2.11) this gives

$$[(\ln|j_0|)_v]_t/f(t)=[(\ln|j_0|)_v]_x/g(x) \tag{2.14}$$

and introducing

$$\frac{d\bar{x}}{dx}=g(x) \qquad \frac{d\bar{t}}{dt}=f(t) \qquad 2u=\bar{x}-\bar{t} \qquad 2v=\bar{x}+\bar{t} \tag{2.15}$$

it is easy to integrate (2.14) to obtain $j_0=l(u)m(v)$, which means that our equation is self-adjoint.

Thus we have our first result:

Equation (1.1) is self-adjoint if and only if

$$c^2(x,t)=f^2(t)/g^2(x). \tag{2.16}$$

If we begin with (1.1), assume (2.11), and do the transformation (2.15) we obtain our second result:

The normal form of (1.1) with $c(x,t)=f(t)/g(x)$ is

$$\{\partial^2_{uv}-[J_1(\bar{x})-\tilde{J}_1(\bar{t})]\}\psi_0=0 \tag{2.17}$$

where

$$J_1(\bar{x})=\frac{(g^{1/2})_{\bar{x}\bar{x}}}{g^{1/2}} \qquad \tilde{J}_1(\bar{t})=\frac{(f^{1/2})_{\bar{t}\bar{t}}}{f^{1/2}}$$
$$W(x,t)=\psi_0(x,t)/\sqrt{g}\sqrt{f}. \tag{2.18}$$

Note that the lower case $j_0(u,v)$, $j_1(u,v)$ refer to the general normal form (1.5), while the alternative normal form (2.17), which exists precisely in the self-adjoint, that is factorised velocity, case, is expressed in terms of the single-variable upper case $J_1(\bar{x})$, $\tilde{J}_1(\bar{t})$, with $j_1(u,v)=J_1(v+u)-\tilde{J}_1(v-u)$.

It remains to find those J_1 and $\tilde{J}_1$, and thus f and g, for which (2.17), and thus the self-adjoint specialisation of (1.1), are exactly solvable.

3. The j-sequence

In this section we briefly review results from [3, 9, 10]. Given any equation in the form (1.5) we generate a sequence $\{j_n(u,v)\}$, $-\infty<n<+\infty$ by

$$\frac{j_{n+1}}{j_n}=\frac{j_n}{j_{n-1}}-\partial^2_{uv}\ln|j_n|. \tag{3.1}$$

If this sequence is 'double-terminating', that is if there exists $N>0$, $M<0$ such that $j_{N+1}=0$ and $1/j_{M-1}=0$, it is shown in [3] that (1.5) is exactly solvable in the progressing wave sense and that the general solution is the sum of

$$\psi_A=\frac{1}{j_1}\partial_v\left(\frac{j_1}{j_2}\right)\ldots\partial_v\left(\frac{j_{N-1}}{j_N}\right)\partial_v(j_N a(v))$$
$$\psi_R=j_{-1}\partial_u\left(\frac{j-2}{j-1}\right)\ldots\partial_u\left(\frac{j_M}{j_{M+1}}\right)\partial_u(j_M b(u)). \tag{3.2}$$

It is clear that by carrying out the indicated differentiations in (3.2) one obtains examples of (1.3), including formulae for the g_n and f_n in terms of $j_M, j_{M+1}, \ldots, j_{N-1}, j_N$. It seems likely that *only* equations generating such double-terminating sequences are exactly solvable, however, we have no proof to that effect, and this is one of several reasons why we cannot be sure we shall ultimately obtain *all* cases of SSAE.

The condition that $j_0(u, v)$ and $j_1(u, v)$ generate a doubly-terminating j-sequence amounts to a nonlinear partial differential equation in two variables of order $2(N+|M|)$. In [10] it was pointed out that this nonlinear problem, which is easily translated into a set of $N+|M|$ coupled nonlinear partial differential equations in two variables, is exactly the same as the dynamical system known as the finite two-dimensional Toda lattice with free ends [11]. Fortunately, the non-trivial general solution for this problem is known [12] and was used in [10] to give a general solution to the condition for double termination of the j-sequences. As we are concerned with self-adjoint equations we need the specialisation of that general result to cases where $j_0(u, v)=1$. This non-trivial specialisation was given in [9] and we restate it here. We note first that substituting $j_0=1$ into (3.1) implies

$$j_n j_{-n}=1 \qquad \forall n \tag{3.3}$$

so it is sufficient to find $j_1, j_2, \ldots, j_N$. We define

$$\underset{\sim}{X}_N=\frac{\Sigma_{n=0}^{2N}(-1)^n I^n(u) I^n(v)}{\varphi_1(u)\ldots\varphi_N(u)\psi_1(v)\ldots\psi_N(v)} \tag{3.4}$$

where $\varphi_1, \ldots, \varphi_N$ are arbitrary functions of u, $\psi_1, \ldots, \psi_N$ are arbitrary functions of v, and

$$I(u)=\overbrace{\int^u \varphi_1 \int \varphi_2 \ldots \int \varphi_2 \int \varphi_1}^{n} \qquad n=0, \ldots, 2N, \tag{3.5}$$
$$I(v)=\overbrace{\int^v \psi_1 \int \psi_2 \ldots \int \psi_2 \int \psi_1}^{n}.$$

In (3.5) if n is even each function appears twice in the integrand, while if n is odd $(2k+1)$, $\varphi_k(\psi_k)$ appears just once, and in the middle spot. We also define a differential operator Δ_n on functions of u and v by

$$\Delta_n \underset{\sim}{X}_N \equiv \det\begin{pmatrix} \underset{\sim}{X}_N & \partial_u \underset{\sim}{X}_N & \cdots & \partial_u^{n-1} \underset{\sim}{X}_N \\ \partial_v \underset{\sim}{X}_N & \partial_{uv} \underset{\sim}{X}_N & \cdots & \partial_u^{n-2}\partial_v \underset{\sim}{X}_N \\ \vdots & \vdots & & \vdots \\ \partial_v^{n-1} \underset{\sim}{X}_N & \partial_v^{n-1}\partial_u \underset{\sim}{X}_N & \cdots & \partial_u^{n-1}\partial_v^{n-1} \underset{\sim}{X}_N \end{pmatrix}. \tag{3.6}$$

Then a doubly-terminating sequence of j compatible with $j_0=1$ and $j_{-n}j_n=1$ and of *total* length $2N+1$ is given by

$$j_n=(-1)^{N-n}\Delta_{N-n}\underset{\sim}{X}_N / \Delta_{N-n+1}\underset{\sim}{X}_N \qquad 1 \leqslant n \leqslant N \tag{3.7}$$

and (3.3). The number of arbitrary functions in the specification of $\underset{\sim}{X}_N$ by (3.4) is such that (3.7) gives a general solution to the problem, however, as we are dealing with a nonlinear system it is not ensured that (3.7) gives all solutions. This is a second reason why we may not be obtaining all the SSAE with this derivation.

4. The solvable equations

We wish to find choices of $J_1(\bar{x})$ and $\tilde{J}_1(\bar{t})$ for which (2.17) is exactly solvable. We first note that if (2.17) is exactly solvable with $\tilde{J}_1(\bar{t})=0$ then the corresponding j_n and so f_n and g_n depend exclusively on $\bar{x}$, and similarly for $J_1(\bar{x})=0$ and $\bar{t}$. We now show [13] that:

If

$$\psi=\sum_{n=0}^{N} a_n(x)(\partial_t)^n p(t\pm x) \tag{4.1}$$

solves

$$\psi_{tt}-\psi_{xx}+G(x)\psi=0 \tag{4.2}$$

while

$$\varphi=\sum_{n'=0}^{N'} b_{n'}(t)(\partial_t)^{n'} q(t\pm x) \tag{4.3}$$

solves

$$\varphi_{tt}-\varphi_{xx}+V(t)\varphi=0 \tag{4.4}$$

then

$$\Psi=\sum_{n'=0}^{N'}\sum_{n=0}^{N} b_{n'}a_n(\partial_t)^{n+n'}F(t\pm x) \tag{4.5}$$

solves

$$\Psi_{tt}-\Psi_{xx}+[G(x)+V(t)]\Psi=0. \tag{4.6}$$

The argument is straightforward. If we substitute (4.5) into (4.6), and use the fact that for each n' (4.1) with $p=(\partial_t)^{n'}F$ solves (4.2), and similarly that for each n (4.3) with $q=(\partial_t)^n F$ solves (4.4), we get

$$\Psi_{tt}-\Psi_{xx}+(G+V)\Psi=\sum_{n'=0}^{N'}\sum_{n=0}^{N} 2[\partial_t a_n)(\partial_t b_{n'})-(\partial_x a_n)(\partial_x b_{n'})](\partial_t)^{n+n'}F. \tag{4.7}$$

But as noted $\partial_t a_n=\partial_x b_{n'}=0$, so each term on the right-hand side of (4.7) vanishes and (4.5) is a (progressing wave) solution of (4.6). This result reduces the problem to a search for double-terminating j-sequences depending on either of $\bar{x}=v+u$ or $\bar{t}=v-u$, but not both. However, a large class of such solutions is readily available. It is sufficient that we choose $\varphi_1,\ldots,\varphi_N$ and $\psi_1,\ldots,\psi_N$ in (3.5) so that the $\underset{\sim}{X}_N$ depend on just one of the $v+u$ or $v-u$, respectively. The first solution follows if we take

$$\varphi_n=c_n\,\mathrm{e}^{K_n u} \qquad \psi_n=d_n\,\mathrm{e}^{K_n v} \tag{4.8}$$

where the c_n, dn and K_n are constants, and in integrating the exponentials in (3.5) we 'choose the constants of integration to vanish'; the second solution is similarly achieved, with

$$\varphi_n=\tilde{c}_n\,\mathrm{e}^{-\tilde{K}_n u} \qquad \psi_n=\tilde{d}_n\,\mathrm{e}^{\tilde{K}_n v}. \tag{4.9}$$

It is not obvious that this is the only way in which to achieve our goal of making $\underset{\sim}{X}_N$ depend on just $v+u$ or just $v-u$; in fact it definitely misses one set of examples which we shall give explicitly below. This is the third reason why we may not be constructing all the SSAE. The missed case is, however, quite special and with its explicit inclusion (4.8) and (4.9) may provide a complete solution to this stage of the construction.

Let us summarise the construction of the SSAE. (i) Substitute (4.8) and (4.9) into (3.5) to obtain, via (3.4), $X_N(v+u)$, $\tilde{X}_N(v-u)$. (ii) Substitute X_N, $\tilde{X}_N$ into (3.6) and (3.7) to obtain $j_{-N}(\bar{x}), \ldots, j_N(\bar{x})$, $\tilde{j}_{-N'}(\bar{t}), \ldots \tilde{j}_{N'}(\bar{t})$. (iii) Substitute these j into (3.2) to obtain ψ_A, ψ_R and $\tilde{\psi}_A$, $\tilde{\psi}_R$, and from these $g_n(\bar{x})$, $\tilde{f}_n(\bar{t})$, of (1.3). (iv) use these functions in (4.5) to obtain the progressing wave general solutions of (2.17), with $J_1(\bar{x})$, $\tilde{J}_1(\bar{t})$ known from step (ii). (iv) Use (2.18) to find $g(\bar{x})$, $f(\bar{t})$, and then (2.15) to find $\bar{t}(t)$, $\bar{x}(x)$, and thus $g[\bar{x}(x)]$, $f[\bar{t}(t)]$, so that you know which examples of (1.1) you have solved exactly.

Fortunately, the literature provides some nicely structured families of examples of $J_1(\bar{x})$ $[\tilde{J}_1(\bar{t})]$, found independently of this involved construction, which generate doubly-terminating j-sequences. it is shown in [14] that if

$$j_1(v+u) = J_1(\bar{x}) = \frac{l(l+1)}{\bar{x}^2} \qquad \left[j_1(v-u) = \tilde{J}_1(\bar{t}) = \frac{l(l+1)}{\bar{t}^2}\right] \tag{4.10}$$

then we have a j-sequence of total length $2l+1$, with $j_0 = 1$. Similarly we must obtain a double termination for

$$\begin{gathered} j_1(v+u) = J_1(\bar{x}) = -\frac{l(l+1)}{b^2 \cosh^2(\bar{x}/b)} \\ \left[j_1(v-u) = \tilde{J}_1(\bar{t}) = -\frac{l(l+1)}{d^2 \cosh^2(\bar{t}/d)}\right] \end{gathered} \tag{4.11}$$

since this ostensibly different set of $j_1(v+u)$ $[j_1(v-u)]$ follows directly from (4.10) by applying the coordinate transformation

$$v = \coth\frac{v'}{b} \qquad u = \tanh\frac{u'}{b} \tag{4.12}$$

to (4.10), and dropping the primes. A more complicated and *inequivalent* set of j_1 resulting in doubly-terminating j-sequences is

$$\begin{gathered} J_1(\bar{x}) = \frac{l'(l'+1)}{b^2 \sinh^2(\bar{x}/b)} - \frac{l(l+1)}{b^2 \cosh^2(\bar{x}/b)} \\ \left[\tilde{J}_1(\bar{t}) = \frac{l'(l'+1)}{b^2 \sinh^2(\bar{t}/b)} - \frac{l(l+1)}{d^2 \cosh^2(\bar{t}/d)}\right]. \end{gathered} \tag{4.13}$$

Other, similar, examples are given in [14]. The examples given by (4.10) are obviously not obtainable via (4.8) or (4.9), while the others are [15].

5. A concluding example

Let us assume that $J_1(\bar{x})$ is given by (4.10). It is known [14] that in that case (2.17) with $\tilde{J}_1(\bar{t}) = 0$ is solved by

$$\psi = \sum_{m=0}^{l} \frac{c_{lm}}{\bar{x}^m}\left(\frac{\mathrm{d}}{\mathrm{d}v}\right)^{l-m} a(v) + \sum^{l} \frac{c_{lm}}{\bar{x}^m}\left(\frac{\mathrm{d}}{\mathrm{d}v}\right)^{l-m} b(u) \qquad c_{lm} \equiv (-1)^m \frac{(l+m)!}{m!(l-m)!} \tag{5.1}$$

which allows us to read off the $g_n(\bar{x})$ and $f_n(\bar{x})$. Thus we have jumped into the construction summarised above at step (iii). Corresponding results for $J_1(\bar{x})=0$ and $\tilde{J}_1(\bar{t})$ given by (4.10) can be obtained by $u\to -u$. It follows from (2.18), $J_1(\bar{x})=(g^{1/2})_{\bar{x}\bar{x}}/g^{1/2}$, that

$$g^{1/2}(\bar{x})=\alpha\bar{x}^{l+1}+\beta\bar{x}^{-l} \tag{5.2}$$

where α and β are arbitrary constants.

If we first consider separately the cases $\alpha=1$, 0 and $\alpha=0$, $\beta=1$ we obtain from (2.15) that

$$x=-1/\bar{x}^{2l+1} \qquad x=\bar{x}^{2l+1} \tag{5.3}$$

respectively, where (5.3) has been simplified by the freedom of a linear transformation on x. Inverting (5.3) and substituting back into (the appropriate specialisations of) (5.2) yields

$$g[\bar{x}(x)]=\bar{x}^{(2l+2)/(2l+1)} \qquad g[\bar{x}(x)]=\bar{x}^{2l/(2l+1)} \tag{5.4}$$

respectively. Using the freedom of linear transformations again we obtain

$$c(x)=(ax+b)^{(2l+2)/(2l+1)} \qquad c(x)=(ax+b)^{2l/(2l+1)} \tag{5.5}$$

$$c(t)=(\tilde{a}t+\tilde{b})^{-(2l+2)/(2l+1)} \qquad c(t)=(\tilde{a}t+\tilde{b})^{-2l/(2l+1)} \tag{5.6}$$

as examples of velocity functions for which (1.1) is exactly solvable. The particular cases corresponding to $l=0$ and $l=1$ are among a few examples of such velocity functions found in [8]; clearly (5.5) and (5.6) are a significant generalisation of those results. Another major generalisation follows from applying (4.1) through (4.6) to conclude that $c(x,t)=c_1(x)c_2(t)$, with $c_1(x)$ either of (5.5) and $c_2(t)$ either of (5.6), also yields an exactly solvable acoustic equation. Finally, a class of examples combining the above arises from the linear combination (5.2). In this case (2.15) can be integrated to obtain

$$x=1/(\alpha\bar{x}^{2l+1}+\beta). \tag{5.7}$$

Inverting (5.7) and substituting back into (5.2) yields

$$g[\bar{x}(x)]=\frac{1}{x^2}\left(\frac{1-\beta x}{\alpha x}\right)^{-2l/(2l+1)} \tag{5.8}$$

which yields

$$c(x)=(ax+b)^2\left(1-\frac{\beta(zx+b)}{\alpha(ax+b)}\right)^{2l/(2l+1)}$$
$$c(t)=\frac{1}{(\tilde{a}t+\tilde{b})^2}\left(\frac{1-\beta(\tilde{a}t+\tilde{b})}{\alpha(\tilde{a}t+b)}\right)^{-2\tilde{l}/(2\tilde{l}+1)} \tag{5.9}$$

along with their product, as velocity functions consistent with exact solvability.

More generally, can we expect all of the integrals that arise in steps (4) and (5) in the above summary to be evaluable in a nice form? In calculating $(g^{1/2})(\bar{x})$ from (2.18) the answer is positive since we are using $J_1(\bar{x})$ for which (2.18) is 'exactly solvable'; the solution can be obtained as a linear combination of ψ_{A} and ψ_{R} of (3.2) with $a(v)$ and $b(u)$ taken to be constants. However, the resulting $g(\bar{x})$ are complicated, and it is not clear that (2.15) can be integrated *and inverted* to find a useful form for $c(x)$, in general. For example, from $J_1(\bar{x}) = 2/\sinh^2 \bar{x}$ we easily obtain that $c(x, t) = \tanh[\bar{x}(x)]$ results in an exactly solvable acoustic equation, but $\bar{x}(x)$ is the inverse of $x = \bar{x} - \tanh \bar{x}$. In situations like this the normal form (1.5), or more specifically (2.17), is the 'right' normal form in which to study the problem.

Let us conclude by considering the non-self-adjoint case, briefly. Since the Kundt-Newman criterion for double termination of the j-sequence has been solved in the general case in [10] it would appear that progress in this case should be possible. Unfortunately, this is not necessarily so. The general acoustic equation is characterised by a single function $c(x, t)$, while the normal form (1.5) contains two such functions. Thus, the general acoustic equation must transform to (1.5) with a restriction on the pair $j_0(u, v)$, $j_1(u, v)$; at this time the form of this restriction is not known. To put it simply, we do not have the analogue of (2.17) for the general case, thus we do not have the class of normal forms, corresponding to (1.1), to which to apply the results of [10]. Efforts in this direction are continuing.

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